Intro to Etale Morphisms

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1 Intro:

The purpose of this lecture is to introduce the notion of an *etale morphism* of schemes, state some of their properties, and give a few examples. There's far more to cover than is possible in a single lecture, so this is necessarily sketchy.

We'll begin by introducing the definitions of smooth, unramified, and etale morphisms from the functor of points perspective, and establish a few basic properties. From this perspective, the fact that etale morphisms organize into a Grothendieck topology will be essentially immediate from the definition. We'll then establish a key motivating property of etale morphisms: they induce isomorphisms on tangent spaces, which is one of many possible justifications for the intuition that etale morphisms are the algebraic analogue of 'local isomorphisms'.

Following this, we'll compare the functorial definitions to more 'algebraic' definitions, and give several examples.

2 The Functorial Perspective:

Recall, any closed immersion of schemes $Z \xrightarrow{i} X$ is induced by a sheaf of ideals $\mathcal{I}_Z$. We'll say a closed immersion $Z \to X$ is a *nilpotent thickening* if $\mathcal{I}_Z$ is a nilpotent ideal. If $\mathcal{I}_Z$ is a square-zero ideal, we'll say $Z \to X$ is a *first-order thickening*.

Definition: A morphism $f : X \to S$ of schemes is said to be *formally etale* (respectively, *formally unramified*, *formally smooth*) if and only if for each diagram

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ T' & \xrightarrow{q} & S \end{array}$$

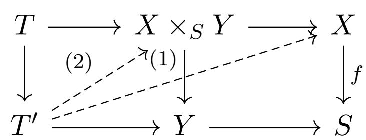
where $T \to T'$ is a first-order thickening, there exists a unique lift of q to X (respectively, if such a lift exists it's unique, there exists a not-necessarily unique lift).

If we impose the additional condition that f be locally of finite type, then formally etale is the same as etale (similarly for smooth and unramified). Looking towards the future, we eventually want to organize certain collections of etale morphisms into a Grothendieck topology. The following properties ensure that these collections verify the definition:

Lemma 2.1. *Formally etale morphisms are stable under base change and composition.*

Proof. There's essentially nothing to check, thanks to our nice definition. We'll verify base change, and leave composition as an exercise.

Fix morphisms $f : X \to S$ and $Y \to S$ such that f is formally etale. We must verify that $X \times_S Y \to Y$ is formally etale. Fix a test diagram

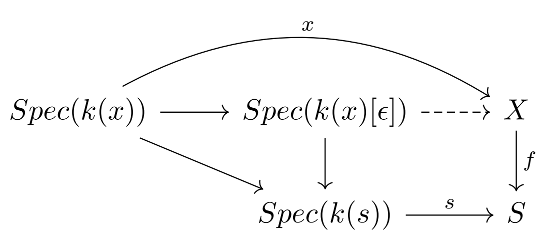


and observe that the lift labelled (1) exists since f is formally etale. Since the rightmost square is a pullback, the first lift induces the lift labelled (2). Uniqueness is obvious.

□

The basic tenet of etale morphisms is that they're the algebraic analogue of local isomorphisms. There are many ways to provide evidence for this analogy, but we'll focus on the differential-geometric motivation for this claim: namely that etale morphisms induce isomorphisms on tangent spaces. We'll begin by recalling the construction of the relative tangent space to an S-scheme.

Given an S-scheme $X \xrightarrow{f} S$ and a point $x \in X$, let $s = f(x)$ and denote by $k(x), k(s)$ their residue fields. For any field k, we'll denote by $k[\epsilon]$ the dual numbers over k (i.e. $k[\epsilon] \simeq k[t]/(t^2)$). Then, *the relative tangent space at* x, denoted $T_{X/S,x}$, consists of those morphisms $Spec(k(x)[\epsilon]) \to X$ fitting into the diagram



This set inherits a canonical $k(x)$-vector space structure. Roughly, this structure is given as follows: for $\theta_1, \theta_2 \in T_{X/S,x}$, we define $\theta_1 + \theta_2 \in T_{X/S,x}$ as the composite

$$Spec(k(x)[\epsilon]) \to Spec(k(x)[\epsilon_1, \epsilon_2]) \xrightarrow{(\theta_1,\theta_2)} X$$

where $k(x)[\epsilon_1, \epsilon_2]$ is given by $\epsilon_i^2 = 0 = \epsilon_1\epsilon_2$ (one can verify that $Spec(k(x)[\epsilon_1, \epsilon_2])$ is the pushout of two copies of $Spec(k(x)[\epsilon])$). Scaling is given by the automorphisms of $Spec(k(x)[\epsilon])$ given by scaling ϵ.

Observe that for any map $f : X \to Y$ of S-schemes, f induces a canonical linear map

$$df : T_{X/S,x} \to T_{Y/S,f(x)} \otimes_{k(f(x))} k(x)$$

where the right-hand tensor product can be identified with those maps $Spec(k(x)[\epsilon]) \to Y$ fitting into the appropriate diagram.

Here, we verify the basic tenet of etale morphisms.

Lemma 2.2. *Let $f : X \to Y$ be a formally etale morphism of S-schemes. Then the differential*

$$df : T_{X/S,x} \to T_{Y/S,f(x)} \otimes_{k(f(x))} k(x)$$

is an isomorphism.

Proof. Observe that any element $\theta \in T_{Y/S,f(x)} \otimes_{k(f(x))} k(x)$ induces a diagram

$$\begin{array}{ccc} Spec(k(x)) & \xrightarrow{x} & X \\ \downarrow & & \downarrow f \\ Spec(k(x)[\epsilon]) & \xrightarrow{\theta} & Y \end{array}$$

Since $Spec(k(x)) \to Spec(k(x)[\epsilon])$ is a first-order thickening, there exists a unique lift of θ to X. So df is bijective.

□

Exercise: Suppose $X \xrightarrow{f} Y \xrightarrow{g} S$ are morphisms of schemes such that $g \circ f$ is formally etale and g is formally unramified. Prove that f is formally etale.

3 Other Definitions and Examples:

One property of etale and unramified morphisms is that they're local (both on the source and the target). This isn't immediately clear given our previous definition, but the definitions we give now make this important property manifest. We begin with the corresponding definitions for commutative rings.

Definition: A map of local rings $f : A \to B$ is said to be *unramified* if

$$f(\mathfrak{m}_A)B = \mathfrak{m}_B,$$

and $k(B)$ is a finite separable extension of $k(A)$.

Any time we're given a property of morphisms of local rings, we can upgrade it to a property of schemes by requiring a morphism of schemes to satisfy the property on stalks. The resulting geometric property is manifestly local.

Definition: A morphism of schemes $f : X \to Y$ is *unramified* if it is locally of finite type and for all $x \in X$, the map of local rings

$$f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is unramified.

Example 1: Consider a quasi-compact morphism $f : X \to Spec(k)$ for some field k. The unramified condition on local rings forces X to be zero-dimensional, which, together with quasi-compactness, forces X to affine. Appealing to Noether normalization, we see that $X = Spec(A)$ for some finite A-algebra, which is thus a product of field extensions of k. The separability condition insures that each such extension is a separable extension, so $A \simeq \prod_{i=1}^{n} k_i$, where k_i is separable.

Example 2: Let L/K be an extension of number fields, and let $\mathcal{O}_L, \mathcal{O}_K$ be the corresponding rings of integers. When is the inclusion $i : \mathcal{O}_K \to \mathcal{O}_L$ unramified? There are two things to check. First, for $\mathfrak{q} \in Spec(\mathcal{O}_L)$, and $\mathfrak{p} := \mathfrak{q} \cap \mathcal{O}_K$, I claim that the extension of residue fields $k(\mathfrak{p}) \subset k(\mathfrak{q})$ is always a finite separable extension. Indeed, since the ring of integers inside a number field is always 1-dimensional, the primes $\mathfrak{p}$ and $\mathfrak{q}$ are either 0 (in which case the claim trivially follows), or maximal, in which case the residue fields can simply be computed as $k(\mathfrak{p}) = \mathcal{O}_K/\mathfrak{p}$ and $k(\mathfrak{q}) = \mathcal{O}_L/\mathfrak{q}$. But the natural map

$$\mathcal{O}_K/\mathfrak{p} \to \mathcal{O}_L/\mathfrak{q}$$

is an integral extension of perfect fields, and thus finite and separable.

Thus, $\mathcal{O}_K \to \mathcal{O}_L$ is unramified if and only if for each $\mathfrak{p} = \mathfrak{q} \cap \mathcal{O}_K$, the map on localizations satisfies

$$\mathfrak{p}\mathcal{O}_{L,\mathfrak{q}} = \mathfrak{q}$$

which is equivalent to requiring that the ramification index of $\mathfrak{p}$ in L is 1. So the algebro-geometric notion of ramification generalizes the notion from algebraic number theory.

There are lots of properties of unramified morphisms that deserve mention here - we'll only mention a couple, without proof. The first connects the notion of unramified morphisms with the differential study of morphisms, and will be of use in verifying that the previous definition agrees with the functorial definition given in section 1.

Lemma: A morphism $f : X \to Y$ is unramified at $x \in X$ if and only if $\Omega^1_{X/Y,x} = 0$.

The next result is a (one of many) strong topological property enjoyed by unramified morphisms.

Lemma: Any section of an unramified morphism is an open immersion. Any section of a separated morphism is a closed immersion. Thus, if $f : X \to Y$ is separated and unramified, any section of f is an isomorphism onto a connected component (provided Y is connected).

The next notion we need is that of *flatness*.

Definition: A map of rings $f : A \to B$ is *flat* if the functor of abelian categories

$$- \otimes_A B : Mod(A) \to Mod(B)$$

is exact.

A key, but easily verified, property is that flatness is a local property of rings. In other words, $A \to B$ is flat if and only if the maps $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ are flat for all primes $\mathfrak{p}, \mathfrak{q}$ satisfying $\mathfrak{p} = f^{-1}(\mathfrak{q})$.

Definition: A morphism of schemes $f : X \to Y$ is *flat* if for all $x \in X$, the map of local rings

$$f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is flat.

We could have equivalently defined a flat morphism to be one such that the pullback functor $f^{\star} : QCoh(Y) \to QCoh(X)$ is an exact functor. We won't say much about flat maps, except to mention a key topological property.

Lemma: Flat morphisms are open.

Finally, we arrive at etale morphisms.

Definition: A morphism of schemes $f : X \to Y$ is *etale* if it is both flat an unramified (so in particular, it's locally of finite type).

We now present some examples. The second example is particularly important, as locally all etale morphisms are of this form.

Example 1: Consider the inclusion $\mathbb{R}[t] \to \mathbb{C}[t]$. I claim the induced map $Spec(\mathbb{C}[t]) \to Spec(\mathbb{R}[t])$ is etale. Clearly, $\mathbb{R}[t] \to \mathbb{C}[t]$ is flat and of finite type, so really all we need to worry about is ramification. At the zero ideal, there's not much to check. The non-zero prime ideals of $\mathbb{C}[t]$ are exactly those of the form $\mathfrak{q} = \langle t - \lambda \rangle$ for some $\lambda \in \mathbb{C}$, and so we split our analysis in to two cases.

The first case is $\lambda \in \mathbb{R}$. In this case $\mathfrak{p} := \mathfrak{q} \cap \mathbb{R}[t] = \langle t - \lambda \rangle$, and thus on the level of localizations, the map $\mathbb{R}[t]_\mathfrak{p} \to \mathbb{C}[t]_\mathfrak{q}$ takes $t - \lambda$ to $t - \lambda$, and thus

$$\mathfrak{p}\mathbb{C}[t]_\mathfrak{q} = \mathfrak{q}$$

proving that the map is unramified at the prime $\mathfrak{q}$.

The second case is $\lambda \notin \mathbb{R}$. Here we see that $\mathfrak{p} = \left\langle (t - \lambda)(t - \overline{\lambda}) \right\rangle$, and so the map on localizations sends the generator of $\mathfrak{p}$ to $(t - \lambda)(t - \overline{\lambda})$ in $\mathbb{C}[t]_\mathfrak{q}$. But in $\mathbb{C}[t]_\mathfrak{q}$, the element $t - \overline{\lambda}$ is invertible, and thus

$$\mathfrak{p}\mathbb{C}[t]_\mathfrak{q} = \mathfrak{q}$$

which proves the claim.

Thus, $Spec(\mathbb{C}[t]) \to Spec(\mathbb{R}[t])$ is etale.

Example 2: (Standard etale morphisms) The previous example is a particularly simple case of a more general class of examples. Locally, every etale morphism is of the form described below.

Let A be a ring, $P(t) \in A[t]$ be a monic polynomial, and define $B = A[t]/\langle P(t) \rangle$. We'll say $P(t)$ is *separable* if $P'(t)$ is a unit in B. The reasoning for the terminology is that P is separable as above if and only if for each prime $\mathfrak{p} \in Spec(A)$, the image $\overline{P}$ in $k(\mathfrak{p})[t]$ is separable in the sense that it has no repeated roots. One can verify this using standard results from field theory.

Notice that as an A-module, B is free of rank $deg(P)$, so in particular, $A \to B$ is flat. Now, for any prime ideal $\mathfrak{p} \in Spec(A)$, we see that $B \otimes_A k(\mathfrak{p}) \simeq k(\mathfrak{p})[t]/\langle \overline{P} \rangle$, where $\overline{P}$. So in particular the fiber of the morphism

$$Spec(B) \to Spec(A)$$

over $\mathfrak{p}$ is a product of extensions of $k(\mathfrak{p})$, and these extensions are separable if and only if P itself is a separable polynomial. Combined with the fact that we can check unramified-ness over each fiber individually, we conclude:

Lemma: With $A, B, P(t)$ as above, the morphism $Spec(B) \to Spec(A)$ is etale if and only if $P(t)$ is a separable polynomial. Moreover, if $b \in B$ is an element such that $P'(t)$ is a unit in the localization B_b, then $Spec(B_b) \to Spec(A)$ is etale. Morphisms of this last type are called **standard etale morphisms**.

Standard etale morphisms are extremely important due to the following classification result:

Theorem: Let Y be a locally Noetherian scheme. Given a morphism $f : X \to Y$ etale in a neighborhood of $x \in X$, then there exist affine open neighborhoods V and U of x and $y = f(x)$ respectively such that

$$f\big|_V : V \to U$$

is a standard etale morphism.